3. Closed sets, closures, and density

1 Motivation

Up to this point, all we have done is define what topologies are, define a way of comparing two topologies, define a method for more easily specifying a topology (as a collection of sets *generated by a basis*), and investigated some simple properties of bases.

At this point, we will start introducing some more interesting definitions and phenomena one might encounter in a topological space, starting with the notions of *closed sets* and *closures*.

Thinking back to some of the motivational concepts from the first lecture, this section will start us on the road to exploring what it means for two sets to be "close" to one another, or what it means for a point to be "close" to a set. We will draw heavily on our intuition about convergent sequences in $\mathbb{R}^n$ when discussing the basic definitions in this section, and so we begin by recalling that definition from calculus/analysis.

Definition 1.1. *A sequence $\{x_n\}_{n=1}^{\infty}$ is said to converge to a point $x \in \mathbb{R}^n$ if for every $\epsilon > 0$ there is a number $N \in \mathbb{N}$ such that $x_n \in B_\epsilon(x)$ for all $n > N$.*

Remark 1.2. It is common to refer to the portion of a sequence $\{x_n\}_{n=1}^{\infty}$ after some index N—that is, the sequence $\{x_n\}_{n=N+1}^{\infty}$—as a *tail* of the sequence. In this language, one would phrase the above definition as "for every $\epsilon > 0$ there is a tail of the sequence inside $B_\epsilon(x)$."

Given what we have established about the topological space $\mathbb{R}^n_{\text{usual}}$ and its standard basis of ϵ-balls, we can see that this is equivalent to saying that there is a tail of the sequence inside any open set containing x; this is because the collection of ϵ-balls forms a basis for the usual topology, and thus given any open set U containing x there is an ϵ such that $x \in B_\epsilon(x) \subseteq U$.

This observation will inform the definition of sequence convergence in a general topological space, and in particular the definition of the "closure" of a set, as we will see shortly.

A note about the order in which we will cover this material: most textbooks on the subject define what it means for a set to be "closed" first, then define closures of sets. We will use the idea of a "closure" as our *a priori* definition, because the idea is more intuitive.

2 Closures

Definition 2.1. *Let $(X, \mathcal{T})$ be a topological space, and let $A \subseteq X$. We define the closure of A in $(X, \mathcal{T})$, which we denote with $\overline{A}$, by:*

$$x \in \overline{A} \text{ if and only if for every open set } U \text{ containing } x,\ U \cap A \neq \emptyset.$$

Or, in symbols:

$$\overline{A} = \{\, x \in X \ : \ \forall U \in \mathcal{T} \text{ such that } x \in U,\ U \cap A \neq \emptyset \,\}.$$

When there can be no confusion about the topological space with respect to which a closure is being considered, we will simply write $\overline{A}$ without specifying "... in $(X, \mathcal{T})$."

In words, $x \in \overline{A}$ if and only if any open set containing x also contains an element of A. After we define what it means for a set to be closed, we will be able to present an alternate way of defining the closure of a set.

Before going any further with examples we examine some elementary properties of closures, the proofs of which use only the definition of closure and absolutely no cleverness or new ideas.

Proposition 2.2. *Let $(X, \mathcal{T})$ be a topological space and let $A, B \subseteq X$. Then:*

1. $A \subseteq \overline{A}$.
2. $\overline{\overline{A}} = \overline{A}$. *(A mathematician interested in using fancy words would say that "taking closures is an idempotent operation".)*
3. $\overline{A \cup B} = \overline{A} \cup \overline{B}$.
4. *If $X \setminus A$ is open, then $\overline{A} = A$.*
5. *Trivially, $\overline{\emptyset} = \emptyset$ and $\overline{X} = X$.*

Proof.

1. Let $x \in A$. Then clearly any open set U that contains x intersects A, since at least $x \in U \cap A$, and therefore $x \in \overline{A}$.
2. By (1), $\overline{A} \subseteq \overline{\overline{A}}$. On the other hand, let $x \in \overline{\overline{A}}$ and let U be an open set containing x. We need to show that $U \cap A \neq \emptyset$. By assumption $U \cap \overline{A} \neq \emptyset$, so there is at least one point y in this intersection. Since U is an open set containing an element y in the closure of A, $U \cap A \neq \emptyset$ by definition of $\overline{A}$.
3. **Exercise.**
4. Assume $X \setminus A$ is open. Again by (1) we know that $A \subseteq \overline{A}$, so it remains to show that $\overline{A} \subseteq A$. Proceeding by contradiction, suppose $x \in \overline{A} \setminus A \subseteq X \setminus A$. Since $X \setminus A$ is an open set containing the element x of $\overline{A}$, by definition of closure we must have that $(X \setminus A) \cap A \neq \emptyset$, which is obviously impossible. This means there can be no points in $\overline{A} \setminus A$.

□

As promised, here are some examples in familiar settings.

Example 2.3.

1. Working in $\mathbb{R}_{\text{usual}}$, the closure of an open interval (a,b) is the corresponding "closed" interval $[a,b]$ (you may be used to calling these sorts of sets "closed intervals", but we have not yet defined what that means in the context of topology).

   To see this, by 2.2.1 we have that $(a,b) \subseteq \overline{(a,b)}$. We first show that $a,b \in \overline{(a,b)}$. Let U be an open set containing a. Then there is an $\epsilon > 0$ such that $B_\epsilon(a) = (a-\epsilon, a+\epsilon) \subseteq U$. Let $\delta = \min\{\epsilon, b-a\}$. Then $a + \frac{\delta}{2} \in U \cap (a,b)$, as required. The proof for b is essentially the same. This establishes that $[a,b] \subseteq \overline{(a,b)}$.

   Finally, if $x \in \mathbb{R} \setminus [a,b]$, the set $(-\infty,a) \cup (b,\infty)$ is an open set containing x and disjoint from (a,b), witnessing that $x \notin \overline{(a,b)}$.

2. Again in $\mathbb{R}_{\text{usual}}$, show the following straightforward facts for any $a < b < c$:

   (a) $\overline{\{a\}} = \{a\}$.

   (b) $\overline{[a,b)} = [a,b]$.

   (c) $\overline{(a,b) \cup (b,c)} = [a,c]$.

   (d) As you might expect from the name "closed interval", $\overline{[a,b]} = [a,b]$.

   (e) Let $A = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}$. Then $\overline{A} = A \cup \{0\}$.
   (Hint: Use the fact that the elements of A, as a sequence in the obvious order, converge to 0 in the sense you are familiar with as defined in the first section.)

   (f) More generally, if $\{x_n\}_{n=1}^{\infty}$ is a sequence that converges to a point $x \in \mathbb{R}$ (in the first-year calculus sense), then $\overline{\{ x_n : n \in \mathbb{N} \}} = \{ x_n : n \in \mathbb{N} \} \cup \{x\}$.
   (By the way, note that $\{ x_n : n \in \mathbb{N} \}$ and $\{x_n\}_{n=1}^{\infty}$ are different objects. The former is a set of numbers, while the latter is an ordered list of numbers or more properly a function.)

3. Generalizing our first example, let $x \in \mathbb{R}^n$ and $\epsilon > 0$. Then the closure of an ϵ-ball $B_\epsilon(x)$ in $\mathbb{R}^n_{\text{usual}}$ is $\{ y \in \mathbb{R}^n : d(x,y) \le \epsilon \}$.

4. It may seem as though any singleton is its own closure, but this need not be true in all topological spaces. For example, let $X = \{0,1\}$ and let $\mathcal{T} = \{\emptyset, X, \{1\}\}$. Then $(X, \mathcal{T})$ is a topological space, and $\overline{\{1\}} = X$. **Check this.**

5. For a less trivial example of the same idea, let $(\mathbb{R}, \mathcal{T}_{\text{ray}})$ be the reals with the ray topology defined in the first set of notes (in which the open sets are $\emptyset$, $\mathbb{R}$, and rays of the form (a,∞)). Then $\overline{\{7\}} = (-\infty, 7]$.

   To see this, first note that if $x \in (7,\infty)$, then $(7,\infty)$ is itself an open set that is disjoint from $\{7\}$, so $x \notin \overline{\{7\}}$. If $x \le 7$, then any open set containing x is of the form $(7-t,\infty)$ for some $t > 0$, and in particular any such set contains 7, showing that $x \in \overline{\{7\}}$.

We will see later in the course that the property "singletons are their own closures" is a very weak example of what is called a "separation property". Topological spaces that do not have this property, like in this and the previous example, are pretty ugly.

6. In $(X, \mathcal{T}_{\text{discrete}})$, for any $A \subseteq X$, $\overline{A} = A$. In other words, every set is its own closure.

7. In $(X, \mathcal{T}_{\text{indiscrete}})$, for any nonempty $A \subseteq X$, $\overline{A} = X$.

8. In the Sorgenfrey line:

   (a) Show that singletons are their own closures. For example, show that $\overline{\{7\}} = \{7\}$.

   (b) For any $a < b \in \mathbb{R}$, show that $\overline{[a,b]} = [a,b]$ and $\overline{[a,b)} = [a,b)$. This latter fact, which in this context says that basic open sets are their own closures, is weird. Keep it in mind for much later in the course.

   (c) On the other hand, $\overline{(a,b)} = [a,b)$. Indeed, first of all if $x \geq b$, then $[x, x+1)$ is an open set containing x and disjoint from (a,b), so $x \notin \overline{(a,b)}$. Second of all if $x < a$, then $[x-1, x+\frac{a-x}{2})$ is an open set containing x and disjoint from (a,b), so $x \notin \overline{(a,b)}$. Finally, $a \in \overline{(a,b)}$ since if U is an open set containing a, then there is an $\epsilon > 0$ such that $[a, a+\epsilon) \subseteq U$, and $[a, a+\epsilon) \cap (a,b) \neq \emptyset$.

   (d) Show that if $A = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}$, then $\overline{A} = A \cup \{0\}$. This is the same as it was in $\mathbb{R}_{\text{usual}}$

   (e) On the other hand, if $B = \left\{ -\frac{1}{n} : n \in \mathbb{N} \right\}$, show that $\overline{B} = B$. Once we have a general topological definition of sequence convergence, we will be able to see that this sequence does not converge in this topological space.

3 Closed sets

In this section we finally introduce the definition we have been tiptoeing around.

Definition 3.1. *A subset A of a topological space X is said to be closed if $X \setminus A$ is open.*

Caution: "Closed" is not the opposite of "open" in the context of topology. A subset of a topological space can be open and not closed, closed and not open, both open and closed, or neither. We will see some examples to illustrate this shortly.

Cautionary note aside, the definition seems simple enough right? Again, here are some observations that follow immediately from this definition and previous definitions in the course, along with De Morgan's laws.

Proposition 3.2. *Let $(X, \mathcal{T})$ be a topological space.*

1. *X and $\emptyset$ are both closed.*

2. *The union of finitely many closed sets is closed.*

3. *An arbitrary intersection of closed sets is closed.*

Proof. The proofs of (1) and (3) are left as exercises.

For (2), let $C_1, \ldots, C_N$ be closed subsets of X. We have to show that

$$X \setminus \bigcup_{k=1}^{N} C_k$$

is open. By De Morgan's laws, we immediately have

$$X \setminus \bigcup_{k=1}^{N} C_k = \bigcap_{k=1}^{N} (X \setminus C_k),$$

which is open since it is the intersection of finitely many open sets. □

As you might suspect from this proposition, or indeed from the definition of a closed set alone, one can completely specify a topology by specifying the closed sets rather than by specifying the open sets as we have been doing thus far. To be more precise, one can "recover" all the open sets in a topology from the closed sets, by taking complements. There are equivalent notions of "basic closed sets", and so on. For example, given a set X we can define the co-finite topology on X equivalently as the topology in which the closed sets are precisely the finite sets. We will not be defining topologies in this way much (if at all) but it is nice to know it can be done, and is sometimes more convenient.

One question that is still lingering: What do closed sets have to do with closures of sets?

Proposition 3.3. *Let $(X, \mathcal{T})$ be a topological space, and let $A \subseteq X$. Then A is closed if and only if $A = \overline{A}$.*

Proof. ($\Rightarrow$) Assume A is closed. We have already shown that $A \subseteq \overline{A}$, so let $x \in X \setminus A$. But then $X \setminus A$ itself is an open set containing x and disjoint from A, so $x \notin \overline{A}$. This shows that $X \setminus A \subseteq X \setminus \overline{A}$, or in other words that $\overline{A} \subseteq A$.

($\Leftarrow$) Assume $A = \overline{A}$. We want to show that $X \setminus A$ is open. Fix $x \in X \setminus A = X \setminus \overline{A}$. Then by definition of $\overline{A}$ there is an open set U containing x and disjoint from A. That is, $x \in U \subseteq X \setminus A$. Therefore $X \setminus A$ is open, as required. □

Another connection between these two concepts is the following, which is often taken as the definition of the closure of a set in contexts where "closed" is defined before "closure".

Proposition 3.4. *Let $(X, \mathcal{T})$ be a topological space, and let $A \subseteq X$. Then $\overline{A}$ is the intersection of all closed subsets of X that contain A.*

Proof. **Exercise.** (See the Big List.) □

Remark 3.5. It should be noted here that there is a notion which is somehow dual to the idea of the closure of a set called the *interior* of a set. You will explore this in a Big List problem.

On to some examples surrounding closed sets. We have done almost all of the work already, so check all of these results yourself.

Example 3.6.

1. In $(X, \mathcal{T}_{\text{indiscrete}})$, the only closed sets are $\emptyset$ and X.

2. In $(X, \mathcal{T}_{\text{discrete}})$, every subset of X is closed.

3. In $\mathbb{R}_{\text{usual}}$, let $a < b$. Then $\{a\}$, $[a, b]$, $[a, \infty)$, and $(-\infty, b]$ are closed. (a, b), $[a, b)$ and $A = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}$ are not closed.

4. In $(X, \mathcal{T}_{\text{co-countable}})$, a subset A of X is closed if and only if A is countable or $A = X$. This topology can just as easily be specified by saying that X is closed and all countable subsets of X are closed.

5. Let X be a set and let $p \in X$. Consider $\mathcal{T}_p$, the particular point topology. A subset A of X is closed in this topology if and only if $p \notin A$.

6. In the Sorgenfrey Line, let $a < b$. Then $[a, b]$ and $[a, b)$ are closed. This should feel weird. Sets of the form $[a, b)$ are *basic open sets* in the Sorgenfrey line, yet they are also closed. Subsets of a topological space that are both open and closed are called *clopen* sets (mathematicians are bad at naming things). Later in the course, when we talk about a property called *connectedness*, we will see that a topological space that has a basis of clopen sets is *highly* disconnected.

   (a, b), $(a, b]$, and $A = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}$ are not closed in this space. $B = \left\{ -\frac{1}{n} : n \in \mathbb{N} \right\}$ is closed, however.

4 Density

By now you should have some intuition saying that the points in the closure of a set A are "close" to A. Phrased the other way around, $\overline{A}$ is the collection of points that A is close to. In this metaphorical framework, a "dense" set is one that is close to everything.

We will not have many applications for these ideas for the moment, but they will become much more important later.

Definition 4.1. *Let $(X, \mathcal{T})$ be a topological space. A subset $D \subseteq X$ is said to be* <u>dense</u> *if $\overline{D} = X$.*

We can immediately rephrase this definition in an equivalent form that talks about open sets.

Proposition 4.2. *Let $(X, \mathcal{T})$ be a topological space, and let $D \subseteq X$. Then the following are equivalent:*

1. *D is dense.*

2. *For every nonempty open $U \subseteq X$, $D \cap U \neq \emptyset$.*

Proof. **Exercise.** (See the Big List.) □

We will henceforth treat both of the above properties as definitions of "dense", and use them interchangeably.

During the first lecture we said that one of the ideas we would try to formalize is the idea of "largeness", and density is one way to that.

Let's see some examples. Most of the proofs are immediate or ones you have actually done before, though you didn't know the words at the time. Some others are left as exercises on the Big List.

Example 4.3.

1. In any topological space $(X, \mathcal{T})$, X is dense in itself.

2. If $(X, \mathcal{T})$ is a topological space, $D \subseteq X$ is dense, and $D \subseteq A$, then A is dense.

3. Working in $\mathbb{R}_{\text{usual}}$:

   (a) The set $\mathbb{Q}$ of rationals is dense. You proved this on the Big List. Notably this means $\mathbb{R}$ has a countable dense subset. Topological spaces with this property are called *separable*. We will investigate this property more later in the course.

   (b) The set $\mathbb{R} \setminus \mathbb{Q}$ of irrationals is dense.

   (c) The set $\mathbb{N}$ is not dense.

   (d) No finite set is dense.

4. $\mathbb{Q}$ is also dense in the Sorgenfrey Line. You also proved this on the Big List. The irrationals are also dense here.

5. In $(\mathbb{R}, \mathcal{T}_{\text{ray}})$, $\mathbb{N}$ is dense. More generally, any subset of $\mathbb{R}$ that is not bounded above is dense.

6. In $(X, \mathcal{T}_{\text{discrete}})$, the only subset of X that is dense is X itself.

7. In $(X, \mathcal{T}_{\text{indiscrete}})$, every nonempty subset of X is dense.

8. Let $p \in X$ and consider the particular point space $(X, \mathcal{T}_p)$. Then $\{p\}$ is dense, but no other singleton is dense. Try to come up with a condition on subsets of X that is equivalent to being dense in this space.

9. If X is infinite, then any infinite subset of X is dense in $(X, \mathcal{T}_{\text{co-finite}})$. If X is uncountable, then any uncountable subset of X is dense in $(X, \mathcal{T}_{\text{co-countable}})$.