10. Orders and ω_1

Contents

1 Motivation

The main reason we are introducing this topic in this course is so that we can define order topologies, and in particular the order topology on ω_1 and $\omega_1 + 1$. These spaces provide the best natural (ie. not contrived) examples of topological spaces with certain combinations of properties. In particular, we will see that $\omega_1 + 1$ is a non-first countable space that is relatively nice with respect to separation properties (unlike $\mathbb{R}_{\text{co-countable}}$ and $\mathbb{R}_{\text{co-finite}}$, our current go-to examples of non-first countable spaces, neither of which are even Hausdorff).

ω_1 is a tricky object to understand, and the sections about it below will contain more detailed proofs than usual. An optimal approach to defining it would involve the definition of ordinals, along with other material that, while interesting, is best left to a set theory course. Instead we will take some shortcuts in order to begin working with ω_1 as quickly as possible.

All of that said, we will still develop the more general notion of a partial order, which is of independent interest. They crop up in many places in mathematics so it will be useful for you to see a detailed introduction to them, but in particular we will need them to define and use Zorn's Lemma during some important proofs later in the course. The progression from partial order, to linear order, to well-order is a natural one, and we will follow it in the notes below.

2 Partial orders

Definition 2.1. *Let $\mathbb{P}$ be a set and let $\leq$ be a binary relation on $\mathbb{P}$. The pair $(\mathbb{P}, \leq)$ is called a partial order (or a partially ordered set, or a poset) provided that $\leq$ has the following three properties.*

1. *Reflexivity: $p \leq p$ for all $p \in \mathbb{P}$.*
2. *Antisymmetry: $p \leq q$ and $q \leq p$ implies $p = q$, for all $p, q \in \mathbb{P}$.*
3. *Transitivity: $p \leq q$ and $q \leq r$ implies $p \leq r$, for all $p, q, r \in \mathbb{P}$.*

Some texts will define strict partial orders before partial orders (Munkres' text does this, for example), but nothing is gained or lost by doing this.

Strictly speaking, a partial order is a set along with a single, fixed relation $\leq$, but we often use variations of this symbol in ways that should be familiar from the example of the real numbers. For example we will say $p < q$ if $p \leq q$ and $p \neq q$, or we might say $p \geq q$ when we mean $q \leq p$. Sometimes we will also use the symbols $\preceq$ and $\prec$, or put a subscript on $\leq$, when we want to talk about two different partial orders at the same time without confusing their respective relations.

Partial orders come up very often in mathematics. There are many with which you are already very familiar as orders, and many with which you are familiar though probably not as orders.

Example 2.2.

1. $(\mathbb{N}, \leq)$, $(\mathbb{Z}, \leq)$, $(\mathbb{Q}, \leq)$, $(\mathbb{R}, \leq)$, where $\leq$ in each case is the usual ordering on real numbers. These are the partial orders, and you are used to thinking about them as orders.

2. The set of English words, when given the usual alphabetical ordering (or "dictionary ordering", as it is sometimes called), is a partial order.

3. $\mathbb{N}$ with the divisibility relation. That is, the relation $\preceq$ defined by $n \preceq m$ if $n|m$. Take a moment to verify that this is a partial order before reading on.

   This partial order is *very* different from the usual ordering on $\mathbb{N}$. In the usual ordering, given any two natural numbers n and m it is always the case that either $n \leq m$ or $m \leq n$. This is no longer the case with the divisibility relation, since for example neither of 5 and 7 divides the other.

   There is a common name for this phenomenon; if $(\mathbb{P}, \leq)$ is partial order and $p, q \in \mathbb{P}$, we say that p and q are comparable if $p \leq q$ or $q \leq p$. If p and q are not comparable they are called incomparable. We will go a little further with these sorts of definitions in the next section.

4. Given a vector space V, the collection of all subspaces of V is a partial order with the subset relation $\subseteq$. We will see many examples of partial orders using the subset relation, and we will usually call this the "inclusion" relation.

5. Given a ring R, the collection of all ideals of R is a partial order with the inclusion relation. (If you do not know what a ring is, feel free to skip this example.) This is of particular relevance since an application of Zorn's Lemma to this partial order is used to prove that every ideal is contained in a maximal ideal, an important result in algebra.

6. Generalizing very much from the previous two examples, given a set X, $(\mathcal{P}(X), \subseteq)$ is a partial order. Check this.

   In particular, note that the antisymmetry of this partial order is something you use in proofs all the time. Every time you prove two sets are equal by showing containment both ways, you are using the antisymmetry of this partial order.

7. Since there are no existential quantifiers in the axioms that define partial orders, any subset of a partial order is again a partial order, using the restriction of the same relation to the subset. That is to say, you cannot "lose" the property of being a partial order by losing elements of the set.

   This is the sense in which the previous example generalizes the two before it. The partial order of vector subspaces of V is simply a subset of the partial order $(\mathcal{P}(V), \subseteq)$. Vector subspaces are just fancy subsets of vector spaces, after all.

(To use a familiar example, this is not true of vector spaces. The axioms that define vector spaces *do* involve existential quantifiers—specifically, closure under the two operations, the existence of an additive identity, and additive inverses—meaning that, *a priori*, not every sub*set* of a vector space is a vector space. This in turn makes the concept of a vector subspace an interesting one. "Sub-partial orders" are not interesting in and of themselves, though sub-orders with certain additional existential properties are often considered, such as the cofinal subsets of directed sets you may have seen in the supplementary notes on nets and filters.)

8. Let $(X, \mathcal{T})$ be a topological space. Then $(\mathcal{T} \setminus \{\emptyset\}, \subseteq)$ is a partial order. It is not strictly necessary to exclude the empty set (it would still be a partial order with the empty set), but it turns out to be more useful this way, as we will see shortly.

   (This example is again a subset of the partial order $(\mathcal{P}(X), \subseteq)$.)

9. If you read my supplementary notes on nets and filters, you came across the concept of a directed set. These are also called directed preorders, since a preorder is a set equipped with a binary relation that is reflexive and transitive, but not necessarily antisymmetric. Preorders are very useful in and of themselves in mathematics.

   Here is a lovely example of a preorder that is not a partial order. Let H be the set of humans on Earth with English names, and for $p, q \in H$ define $p \leq q$ if and only if p's name is alphabetically before q's name. This partial order is obviously reflexive and transitive, but the statement that this partial order is antisymmetric amounts to "any two people with the same name are the same person".

   Any preorder $(\mathbb{P}, \leq)$ can be "turned into" a partial order by defining the equivalence relation $p \sim q$ if and only if $p \leq q$ and $q \leq p$, then modding out by it to form $(\mathbb{P}/\sim, \leq)$, where $\leq$ here is the induced relation on equivalence classes. (If you have no idea what that means, don't worry about it.)

10. Let $X = \mathbb{N} \cup \{\omega\}$, where ω is just a symbol that is not a natural number. (ω is the lowercase Greek letter omega, not the English letter "w".) Define a relation $\leq$ on X, extending the usual order on the natural numbers, by declaring that $n < \omega$ for all $n \in \mathbb{N}$. ω is essentially a "point at infinity" here. We will refer to this partial order several times. It is usually referred to as $\omega + 1$ (read "omega plus one"). This order has an important relationship with sequences, as we will see shortly.

3 Some terminology

In Example 2.2.3 above, we learned some new terminology. For the sake of clarity and ease of reference, we will formally define this terminology here, along with some other useful terms. This will also allow us to [sort of] explain how the countable chain condition got its name.

Throughout all of these definitions, let $(\mathbb{P}, \leq)$ be a partial order.

Definition 3.1. *Two elements $p, q \in \mathbb{P}$ are called comparable if $p \leq q$ or $q \leq p$. If they are not comparable they are called incomparable.*

Definition 3.2. *A subset $A \subseteq \mathbb{P}$ is called a chain if every pair of elements of A are comparable.*

So in $(\mathbb{R}, \leq)$, the whole partial order is a chain. In $\mathbb{N}$ with the divisibility relation, the set $\{7, 7^2, 7^3, 7^4, \dots\}$ is a chain, but the set of all odd numbers, for example, is not.

Definition 3.3. *Two elements $p, q \in \mathbb{P}$ are called compatible if there is an element $r \in \mathbb{P}$ such that $r \leq p$ and $r \leq q$. If they are not compatible they are called incompatible.*

Notice that comparability is strictly stronger than compatibility. Indeed, if $p \leq q$, then reflexivity of the partial order relation guarantees that p witnesses that p and q are compatible. It is usually more useful to think about this relationship in the contrapositive: incompatibility is stronger than incomparability.

To see that this relationship is strict, notice that 9 and 15 are not comparable in the divisibility order on $\mathbb{N}$ since neither of them divides the other, but they are compatible since 3 divides both of them.

Definition 3.4. *A subset $A \subseteq \mathbb{P}$ is called an antichain if every pair of elements of A are incompatible.*

So the set P of prime numbers is an antichain in the divisibility order on $\mathbb{N} \setminus \{1\}$, since given any two distinct primes, no number in this set divides both of them.

Finally, as promised:

Definition 3.5. $(\mathbb{P}, \leq)$ *is said to have the countable chain condition (or ccc) if every antichain in $\mathbb{P}$ is countable.*

As you can see, this should probably be called the "countable antichain condition", but the name has stuck. This definition is more "fundamental" to partial orders than it is to topologies.

The connection between the two is that a topological space $(X, \mathcal{T})$ has the (topological version of the) ccc if and only if the poset $(\mathcal{T} \setminus \{\emptyset\}, \subseteq)$ we saw earlier has the (partial order version of the) ccc. This is the reason we choose to remove the empty set when defining this partial order, since if it were present it would trivially witness the compatibility of any two open sets.

4 Linear orders

We spent some time talking about general partial orders above—and those concepts will be useful to us later—but linear orders are much more important to us in topology, because they come pre-packaged with topologies in a natural way.

Linear orders will feel much more familiar than partial orders. Our intuition about orders says that any two elements of an order should be comparable, like is true in the real numbers for example.

Definition 4.1. *A partial order $(L, \leq)$ is called a linear order if every two elements of L are comparable. That is, for every $a, b \in L$, either $a \leq b$ or $b \leq a$.*

Using our terminology from the previous section, a partial order is a linear order exactly when the whole set is a chain.

Example 4.2.

1. Again, the familiar orders on real numbers $(\mathbb{N}, \leq)$, $(\mathbb{Z}, \leq)$, $(\mathbb{Q}, \leq)$, $(\mathbb{R}, \leq)$ are all linear orders.

2. $\omega + 1$ is a linear order. To define it we started with the linear order $(\mathbb{N}, \leq)$, and added an element that we defined to be comparable to everything.

3. The dictionary order on English words is a linear order, since given two words one is always alphabetically before the other.

4. $\mathbb{N}$ with the divisibility relation is not a linear order, since we have already seen that two numbers can be incomparable in this ordering.

5. Let $(L_1, \leq_1)$ and $(L_2, \leq_2)$ be two linear orders. Define the lexicographical order $\leq$ on $L_1 \times L_2$ by:

$$(a_1, a_2) \leq (b_1, b_2) \quad \text{if and only if} \quad a_1 <_1 b_1 \quad \text{or} \quad a_1 = b_1 \text{ and } a_2 \leq_2 b_2.$$

   You should recognize this as the way words are ordered alphabetically. To be clear, if A is the English alphabet and $\preceq$ is the alphabetical ordering of letters, then $(A, \preceq)$ is a linear order, and the lexicographical order on $A \times A$ defined as above restricts to the usual way of ordering two-letter words in the dictionary. (That, of course, is where the name "lexicographical order" came from.)

Your favourite linear order is probably the real numbers. Your favourite topology is probably also the usual topology on the real numbers. These two things are intimately connected. The topology on the real numbers is defined *in terms of* the usual ordering on the real numbers. In this same way, we can define a topology on any linear order.

First, some notation. These pieces of notation mirror the notation for intervals in the real numbers.

Let $(L, \leq)$ be a linear order, and let $a, b \in L$. We define the following notation:

- $(a, \infty) = \{\, x \in L : a < x \,\}$.

- $(-\infty, b) = \{\, x \in L \,:\, x < b \,\}$.
- $(a, b) = \{\, x \in L \,:\, a < x < b \,\}$.
- $(a, b] = \{\, x \in L \,:\, a < x \le b \,\}$.

(And so on.)

Definition 4.3. *Let $(L, \le)$ be a linear order with at least two elements. Define the set*

$$\mathcal{S} = \{\, (-\infty, b) \,:\, b \in L \,\} \cup \{\, (a, \infty) \,:\, a \in L \,\}\,.$$

Then $\mathcal{S}$ is a subbasis on L, and the topology it generates is called the order topology on L.

This essentially (but not precisely) means that the order topology on L is the one generated by the basis of open intervals, in the sense described above, since for any $a < b \in L$,

$$(a, b) = (-\infty, b) \cap (a, \infty)$$

and therefore (a, b) is in the basis generated by $\mathcal{S}$. This is just the same as happens in $\mathbb{R}_{\text{usual}}$.

Before going any further, we establish some simple examples.

Example 4.4.

1. The order topology on $(\mathbb{R}, \le)$ is the same as the usual topology. We already know this, since the usual topology is generated by the basis of open intervals.

2. The order topology on $(\mathbb{Q}, \le)$ is the same as its subspace topology inherited from $\mathbb{R}_{\text{usual}}$. This is easy to see, but not trivial, since for example $(-\pi, \pi) \cap \mathbb{Q}$ is an open set in the subspace topology inherited from $\mathbb{R}_{\text{usual}}$, but is not a basic open interval in the order topology on $\mathbb{Q}$. It is, of course, a union of basic open intervals.

3. The order topology on $(\mathbb{N}, \le)$ is also the same as its subspace topology inherited from $\mathbb{R}_{\text{usual}}$, which is to say that it is discrete. Indeed, for example, we have that $\{7\} = (6, 8)$.

   (Remember that here we are talking about intervals *in the linear order*. So this is true since 7 is the only element of $\mathbb{N}$ strictly above 6 and strictly below 8.)

4. Consider the unit square $[0, 1] \times [0, 1]$. One way of looking at this set is as the Cartesian product of two linear orders, and thus a topology on it can be obtained by using the order topology given by the lexicographic ordering on the product, as described in Example 4.2.5.

   This topology is *extremely* different from the subspace topology induced from $\mathbb{R}^2_{\text{usual}}$. We will explore this a bit on the Big List.

Now some examples to illustrate the importance of $\omega + 1$.

Example 4.5.

1. First, a reminder of an example from earlier in the course. Let $X = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\} \cup \{0\}$, and let $Y = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\} \cup \{-7\}$. As we have seen in previous sections of the lecture notes, Y is a discrete subspace of $\mathbb{R}_{\text{usual}}$. X, however, is not a discrete subspace; we know that any open interval in $\mathbb{R}$ containing 0 also contains a tail of the sequence of $\frac{1}{n}$'s, and therefore $\{0\}$ is not open in the subspace topology on X.

   In particular, X and Y are not homeomorphic as subspaces of $\mathbb{R}_{\text{usual}}$.

2. Now consider the same sets X and Y but with their *order* topologies. They are both, after all, linear orders (when equipped with the usual ordering on real numbers).

   In both spaces with their order topologies we still have that all the $\frac{1}{n}$ are open as singletons, since $\{\frac{1}{n}\} = (\frac{1}{n+1}, \frac{1}{n-1})$.

   However, in both of these spaces, the remaining point is not open. In X, the only basic open sets containing 0 are of the form $(-\infty, b) = [0, b)$ for some $b \in \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}$, and any such set contains all but finitely many elements of X. The same is true in Y, since again the only open sets containing -7 must be of the form $(-\infty, b) = [-7, b)$ for some $b \in \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}$.

   The proper way to describe this situation is that in both spaces, the sequence $\left\{ \frac{1}{n} : n \in \mathbb{N} \right\}$ converges to the lower point. The order topology does not care that in Y we "moved" one point over by seven units. It only cares about where the element falls in the order on the set. With their order topologies, these two spaces are homeomorphic.

   Both of these spaces are homeomorphic to the set X with its subspace topology from $\mathbb{R}_{\text{usual}}$ (via the identity function from X to itself). We have a better way of describing this space, however. This is illustrated by the next example.

3. Consider $\omega + 1$ with its order topology. $\{n\}$ is open for all $n \in \mathbb{N}$ for the same reason this was true in $\mathbb{N}$ with its order topology. However in this space we have the extra element ω above everything. A basic open set containing ω must be of the form $(a, \infty) = (a, \omega]$ for some $a \in \mathbb{N}$, and therefore any such set contains all but finitely many members of $\mathbb{N}$.

   We conclude that in $\omega + 1$, the sequence $1, 2, 3, 4, \ldots$ converges to ω (as does any strictly increasing sequence in $\mathbb{N}$). Furthermore, $\omega + 1$ is homeomorphic to the spaces X and Y with their order topologies from the previous example.

   This is why $\omega + 1$ is so important. It is a "model" space that is homeomorphic to the subspace topology on a convergent sequence along with its limit point (in any sufficiently separative topological space).

Order topologies are particularly useful for us to study because they can be a little weird or counterintuitive, but they are still very well behaved with respect to separation, as the following proposition illustrates.

Proposition 4.6. *A linear order with its order topology is Hausdorff, T_3 and T_4.*

(Actually it turns out that linear orders are completely normal, or equivalently hereditarily normal.)

Proof. We first show that linear orders are Hausdorff. Suppose $(L, \leq)$ is a linear order with its order topology, and let p and q be distinct elements of L, and assume without loss of generality that $p \leq q$. If there exists an element a such that $p < a < q$, then $(-\infty, a)$ and (a, ∞) separate the two points, and we are done. If there is no such a, then $(-\infty, q) = (-\infty, p]$ and $(p, \infty) = [q, \infty)$ are disjoint open sets containing p and q, respectively.

We next establish regularity. Suppose $C \subseteq L$ is a closed set and $p \in L \setminus C$. Then there is an open interval (a, b) such that $p \in (a, b) \subseteq L \setminus C$ (since the open intervals are a basis for the order topology).

From this point onward we will use a similar argument to the previous one. If there are elements $u, v \in L$ such that $a < u < p < v < b$, then the open sets $(-\infty, u) \cup (v, \infty)$ and (u, v) are disjoint and contain C and p, respectively.

If there is no such u and/or no such v, we can easily adjust our sets in a similar way as above. For example if no such u exists, we can use $(-\infty, p) \cup (v, \infty) = (-\infty, a] \cup (v, \infty)$ and $(a, v) = [p, v)$, and so on.

The proof that linear orders are normal is tedious and not particularly instructive, so we will omit it from these notes. You are welcome to look into it yourself if you are curious, but I promise you will not be missing anything if you don't. One interesting thing about this proof is that it necessarily uses the Axiom of Choice. Without Choice, there are linear orders that are not normal as topological spaces. □

5 Well-orders

Now we move on to a particularly nice and well-behaved class of linear orders. These types of orders are fundamental to the structure of mathematics (the Axiom of Foundation is essentially a statement about well-orders). They also have an intimate connection to induction, in the sense that they allow us to generalize induction to larger contexts. (We will not be generalizing induction in this course, except possibly on the Big List...)

Definition 5.1. *A partial order $(W, \leq)$ is said to be a well-order if every nonempty subset of W has a $\leq$-least element. That is, for every nonempty $S \subseteq W$ there is an $a \in S$ such that $a \leq x$ for all $x \in S$.*

We will often just say "least" rather than "$\le$-least", when the order relation is clear from context. Also, given a subset S of a well-order W, we will denote its least element by $\min(S)$.

Example 5.2.

1. $(\mathbb{N}, \le)$ is a well-order.
2. $(\mathbb{R}, \le)$ is not a well-order, since $\mathbb{R}$ itself has no least element. Similarly $\mathbb{Q}$ and $\mathbb{Z}$ with their usual orders are not well-orders.
3. $\omega + 1$ is a well-order. Make sure you believe this before reading on.
4. Given two well orders $(W_1, \le_1)$ and $(W_2, \le_2)$, the lexicographical order on their product $W_1 \times W_2$ is a well-order. Check this as well.
5. Every subset of a well-order is again a well-order, under the restriction of the same relation. This is not immediate, as in the case of partial or linear orders, because the definition of a well-order does involve existential quantifiers. It is true though, since a subset of a subset of a well-order $(W, \le)$ is itself a subset of W.

Here are some elementary facts about well-orders. For the duration, let $(W, \le)$ be a well-order.

- Every well-order is a linear order. Indeed, if $p, q \in W$ are distinct elements, then the subset $\{p, q\} \subseteq W$ must have a least element, from which it follows that p and q are comparable.
- Every nonempty $S \subseteq W$ has a *unique* least element.
- There is a least element, $\min(W)$, of W (this is immediate).
- If W has two or more elements, there is a *second least* element of W, which is the least element of $W \setminus \{\min(W)\}$.
- If W has three or more elements, there is a *third least* element of W, as should now be clear.
- If W is infinite, we can proceed inductively as above and find a copy of the well order $\mathbb{N}$ at the "bottom" of W.
- A linear order $(W, \le)$ is a well-order if and only if it contains no infinite, decreasing chains. Proving this will be an exercise on the Big List.
- Since every subset of a well-order has a least element, every nonempty interval (a, b) in a well-order is really of the form $[c, b)$, where $c = \min(a, b)$.

With this definition established and explored a bit, we are ready to move into weird territory.

6 ω_1

ω_1 is a specific well-order that is quite useful to study in many contexts. The definition of this space is going to be unusual. There are many approaches to defining it, some more rigorous than others. A proper treatment would involve defining ordinals, which is more work than we need to do for a topology class. Instead, we are going to introduce it in a way that will not seem well-defined at first (though, of course, it is a well-defined object).

In particular, we are going to work with ω_1 via the properties it satisfies only. We are not going to explicitly "construct" it. This will also probably seem weird at first.

We are going to define ω_1 as a particular subset of an uncountable well-order. At the moment, you do not have any evidence that such a thing even exists; the only well-orders we have explicitly mentioned thus far are countable. The existence of such an order is something we are going to take as axiomatic, because ultimately it *is* axiomatic in mathematics, via the following fact.

Theorem 6.1 (Well-Ordering Principle)**.** *Let X be a set. Then there exists a binary relation $\leq$ on X such that $(X, \leq)$ is a well-order.*

In other words, *any set can be well-ordered.* This theorem is actually *equivalent* to the Axiom of Choice, as we will see soon in this course. So, for the moment, we will not prove it.

Note that this result does not say that *any* order or any linear order is a well-order. Just that every set can be well-ordered. This should strike you as *highly* counterintuitive. For example, you almost certainly cannot imagine a well-ordering of $\mathbb{R}$, or weirder yet of $\mathbb{R}^2$.

Corollary 6.2. *There are uncountable well-orders.*

Having established that there are uncountable well-orders, we are going to fix one of them and define ω_1 to be a particular subset of it.

Definition 6.3. *Let $(W, \leq)$ be an uncountable well-order. For each $x \in W$, define the set of* *<u>predecessors</u>* *of x in W in the natural way:*

$$\text{pred}(x) := \{\, y \in W \,:\, y < x \,\}.$$

Then, define

$$\omega_1 = \{\, \alpha \in W \,:\, \text{pred}(\alpha) \text{ is countable} \,\}.$$

I know that this definition looks "non-canonical", in the sense that it seems like if we start with different uncountable well-orders, we might get different objects. That is true in a superficial sense, but no matter which well-order you start with, the "version" of ω_1 you define will be the same *as an order.* Since we only care about it as an order, there will be no problem for us. (In a more set theoretic context ω_1 is explicitly constructed, so there is no such ambiguity.)

From this definition alone we are going to point out some simple properties of ω_1, and then you should always think of it as the well-order that has these properties. They will do all the work. You will rarely have to refer to the definition specifically.

Proposition 6.4. *Some facts about ω_1.*

1. *ω_1, equipped with the restricted ordering from W, is a well-order.*

2. *ω_1 is uncountable.*

3. *ω_1 is closed downwards. That is, for every $\alpha \in \omega_1$, $\text{pred}(\alpha) \subseteq \omega_1$.*

4. *For all $\alpha \in \omega_1$, $\text{pred}(\alpha)$ is countable.*

5. ***(Key property)** Every countable subset of ω_1 is bounded above. That is, if $S \subseteq \omega_1$ is countable, then there is an $\alpha \in \omega_1$ such that $\beta \leq \alpha$ for all $\beta \in S$.*

Proof.

1. This follows from Example 5.2.5.

2. Suppose for the sake of contradiction that ω_1 is countable. By assumption the well-order $(W, \leq)$ is uncountable, and so $W \setminus \omega_1$ is nonempty, and therefore has a least element. Call that element $x_0 := \min(W \setminus \omega_1)$.

   $x_0 \notin \omega_1$, and therefore by definition of ω_1 it must be that $\text{pred}(x_0)$ is uncountable. However, it is easy to see that $\text{pred}(x_0) = \omega_1$, by the minimality of x_0. If ω_1 is countable then this set is certainly also countable, a contradiction.

3. If $\alpha \in \omega_1$ and $\beta \in \text{pred}(\alpha)$, then $\text{pred}(\beta) \subseteq \text{pred}(\alpha)$ by the transitivity of the ordering, and therefore $\text{pred}(\beta)$ is countable.

4. This is immediate from the definition of ω_1. (I just wanted to list it with the other properties, for ease of reference.)

5. Suppose $S \subseteq \omega_1$ is countable and nonempty. By definition of ω_1, $\text{pred}(s)$ is countable for each $s \in S$. Then

$$S' := S \cup \bigcup_{s \in S} \text{pred}(s)$$

   is countable, being a countable union of countable sets. Therefore we can find an element $\alpha \in \omega_1 \setminus S'$, and this α is above every element of S (if it was not above some $s \in S$ it would be an element of $\text{pred}(s)$, which it cannot be).

□

These few properties, particularly the last one, encapsulate most of what we will need to know about ω_1 as a well-order.

From this point forward, we will consider ω_1 as a topological space with its order topology, and analyze some of its properties. Here are a few of them.

Proposition 6.5. *ω_1 is Hausdorff, T_3, T_4. (In fact, it is completely normal.)*

Proof. ω_1 is an order topology, so this just follows from Proposition 4.6. □

Proposition 6.6. *ω_1 is not separable (and therefore not second countable).*

Proof. This follows almost immediately from Fact 6.4.5. Suppose $D \subseteq \omega_1$ is countable. Then D is bounded above, so let $\alpha \in \omega_1$ be above every element of D. But then (α, ∞) is a nonempty open subset of ω_1 that contains no element of D. Therefore, D is not dense. □

Proposition 6.7. *ω_1 is first countable.*

Proof. Fix $\alpha \in \omega_1$. Notice that for $x \in \text{pred}(\alpha)$, the set $(x, \alpha]$ is a basic open set. Indeed, $(x, \alpha] = (x, \alpha + 1)$, where $\alpha + 1$ is common notation for the "next least" element after α. Specifically, $\alpha + 1 := \min(\omega_1 \setminus (\text{pred}(\alpha) \cup \{\alpha\}))$

Now $\mathcal{B}_\alpha = \{\, (x, \alpha] : x \in \text{pred}(\alpha) \,\}$ is a local basis at α, and it is countable since $\text{pred}(\alpha)$ is countable. □

These results alone should convince you that ω_1 is a worthwhile object for us to study. We have not had many interesting examples of non-separable spaces. We also have not had any examples of spaces that are first countable but not second countable other than the Sorgenfrey line. This space is very well-behaved from a separation standpoint, being T_4 (and actually hereditarily T_4), but relatively poorly behaved from the point of view of countability properties.

Here are some other topological facts, whose proofs will be left as exercises on the Big List:

1. ω_1 is not discrete (this may not be obvious at this point).
2. ω_1 does not have the countable chain condition.
3. ω_1 is not homeomorphic to any subspace of $\mathbb{R}_{\text{usual}}$. In fact, any continuous function $f : \omega_1 \to \mathbb{R}$ is eventually constant, and so there are not even any continuous bijections here.

7 $\omega_1 + 1$

Earlier we learned about the order $\omega + 1$, which we obtained by taking the usual order on the natural numbers and adding a new element above everything. If we do the same thing to ω_1, we get $\omega_1 + 1$, whose formal definition is as follows.

Definition 7.1. *Let Ω be a symbol that is not an element of ω_1. Then we define $\omega_1 + 1 := \omega_1 \cup \{\Omega\}$. We define an ordering on $\omega_1 + 1$ by extending the ordering on ω_1, declaring that $\alpha < \Omega$ for all $\alpha \in \omega_1$.*

$\omega_1 + 1$, as a topological space with its order topology, is a great one. Here are some quickly properties of this space.

Proposition 7.2. *$\omega_1 + 1$ is Hausdorff, T_3, T_4. (In fact, it is completely normal.)*

Proof. It is an order topology, so this again just follows from Proposition 4.6. □

Proposition 7.3. *$\omega_1 + 1$ is not separable (and therefore not second countable).*

Proof. The same basic idea as in the proof of Proposition 6.6 works here with a minor modification. If $S \subseteq \omega_1 + 1$ is a countable set that does not contain Ω, the proof is identical.

If S does contain Ω, let $S' = S \setminus \{\Omega\}$—which is also countable—and let α be an upper bound for S' in ω_1. The set (α, ∞) used in the earlier proof is replaced by (α, Ω), and this open set contains no element of S. □

This next proposition is the real payoff here. It is the reason we bother to define $\omega_1 + 1$ in this course.

Proposition 7.4. *$\omega_1 + 1$ is **not** first countable.*

Proof. All the elements of ω_1 have the same countable local bases as in the proof of Proposition 6.7. So our only hope is to show that Ω does not have a countable local basis. We will actually show that there is no countable local basis consisting of basic open sets, which will of course suffice to prove the result. Note that any basic open set containing Ω must be of the form $(\alpha, \Omega]$ for some $\alpha \in \omega_1$.

Suppose $\mathcal{B} = \{ (\alpha_n, \Omega] : n \in \mathbb{N} \}$ is a countable collection of basic open sets containing Ω. We want to show that $\mathcal{B}$ is not a local basis at Ω.

The set $S = \{ \alpha_n : n \in \mathbb{N} \} \subseteq \omega_1$ is countable, and therefore bounded above. Let $\alpha \in \omega_1$ be above every element of S. Then $(\alpha, \Omega]$ is a basic open set containing Ω, but which contains no element of $\mathcal{B}$ as a subset. Therefore, $\mathcal{B}$ is not a local basis at Ω. □

So there we have it. $\omega_1 + 1$ is the first interesting non-first countable space we have been able to define. Our original examples of a non-first countable spaces were $\mathbb{R}_{\text{co-countable}}$ and $\mathbb{R}_{\text{co-finite}}$, which are both *highly* pathological spaces. They are not even Hausdorff. $\omega_1 + 1$ on the other hand is very well-behaved from a separation standpoint, being normal, and relatively natural to define (if you are already thinking about order topologies). It also does not feel as though it was invented specifically to break first countability, the way the Arens-Fort space that many of you discovered while doing an earlier Big List problem seems to be.