13. Urysohn's Lemma

1 Motivation

Urysohn's Lemma (it should really be called Urysohn's *Theorem*) is an important tool in topology. It will be a crucial tool for proving Urysohn's metrization theorem later in the course, a theorem that provides conditions that imply a topological space is metrizable. Having just learned about metrizable spaces and how nice they are, that alone should seem worthwhile.

This theorem furnishes us with an alternative way of characterizing normality, which you will recall is a separation property concerned with separating closed sets with open sets. In section 9 of the Big List, you studied various separation properties related to Hausdorff, regularity and normality that were defined in terms of continuous functions. You learned what it means to separate points and sets with a continuous function rather than with open sets, and shortly thereafter learned that being able to separate points with continuous functions is stronger than being able to separate points with open sets (in the language we introduced, you learned that "completely Hausdorff" is strictly stronger than "Hausdorff", and that "completely regular" is strictly stronger than "regular").

Urysohn's Lemma is the surprising fact that being able to separate closed sets from one another with a continuous function *is not* stronger than being able to separate them with open sets.

This theorem is the first "hard" result we will tackle in this course. The proof is truly novel and interesting, involving a very clever construction of a continuous function.

2 Preliminaries

All of these things have appeared in previous sections of the notes. They appear here as reminders.

Definition 2.1. *A topological $(X, \mathcal{T})$ space is called* <u>*normal*</u> *if for every pair of disjoint nonempty closed subsets $C, D \subseteq X$ there exist disjoint open sets U, V such that $C \subseteq U$ and $D \subseteq V$.*

Recall the following alternative characterization of this property, which you proved on the Big List. In the proof, we will work entirely with this characterization rather than the usual one.

Proposition 2.2. *A topological space $(X, \mathcal{T})$ is normal if and only if for every open set U and every closed $C \subseteq U$, there is an open set V such that $C \subseteq V \subseteq \overline{V} \subseteq U$.*

We will also need the fact that $\mathbb{Q}$ is countable, and the Completeness Axiom which says for example that any nonempty subset of real numbers that is bounded below has a greatest lower bound.

3 The theorem and the idea of the proof

In this section we formally state the theorem we are going to prove, take care of the easy direction of the proof, and "start" the proof of the other direction in a very concrete way so we can more easily see what is going on in the more complicated proof that follows.

Theorem 3.1 (Urysohn's Lemma)**.** *A topological space $(X, \mathcal{T})$ is normal if and only if for every pair of disjoint nonempty closed subsets $C, D \subseteq X$ there is a continuous function $f : X \to [0, 1]$ such that $f(x) = 0$ for all $x \in C$ and $f(x) = 1$ for all $x \in D$.*

Proof of ($\Leftarrow$). This direction is very easy. Suppose $C, D \subseteq X$ are disjoint nonempty closed sets. Let $f : X \to [0, 1]$ be a continuous function as in the statement of the theorem. Then $C \subseteq f^{-1}([0, \frac{1}{2}))$ and $D \subseteq f^{-1}((\frac{1}{2}, 1])$. These two preimages are disjoint since they are preimages of disjoint sets, and are open since f is continuous. □

The ($\Rightarrow$) direction is the interesting (and difficult) one. For this direction we must *construct* a continuous function from scratch, so to speak. What follows is an example of one way that the initial steps of the proof might go. Instead of continuing it, we will stop and look at what we have after a few steps.

Suppose $(X, \mathcal{T})$ is normal, and fix two disjoint nonempty closed sets $C, D \subseteq X$. Let $U_1 = X$. Now find an open set U_0 such that

$$C \subseteq U_0 \subseteq \overline{U_0} \subseteq X \setminus D,$$

which we can do since $X \setminus D$ is open and $(X, \mathcal{T})$ is normal. Next, find an open set $U_{\frac{1}{2}}$ such that

$$C \subseteq U_0 \subseteq \overline{U_0} \subseteq U_{\frac{1}{2}} \subseteq \overline{U_{\frac{1}{2}}} \subseteq X \setminus D$$

which we can again do since $(X, \mathcal{T})$ is normal. We stop here for the moment. We have now created three open sets: $U_0 \subseteq U_{\frac{1}{2}} \subseteq U_1$, and we have:

$$C \subseteq U_0 \subseteq U_{\frac{1}{2}} \subseteq X \setminus D \subseteq U_1 = X.$$

From these open sets, we can define a function $g : X \to [0, 1]$ by $g(x) = \min\{\, p : x \in U_p \,\}$. This function only takes on the values 0, $\frac{1}{2}$, or 1. We can see from the line of containments above that $g(x) = 0$ for all $x \in C$ since $C \subseteq U_0$, and that $g(x) = 1$ for all $x \in D$ since $D \subseteq U_1$ but both of the other two open sets live in the complement of D. This function, of course, is not necessarily continuous (since for example the preimage of $(\frac{1}{4}, \frac{3}{4})$ is $U_{\frac{1}{2}} \setminus U_0$, which is not necessarily open).

The real proof uses essentially the same idea, but uses *many* more U's to force the function to be continuous.

4 The proof

Suppose $(X, \mathcal{T})$ is normal, and let $C, D \subseteq X$ be disjoint nonempty closed sets. We will inductively construct a collection of open subsets of X indexed by rational numbers in $[0,1]$. That is, we will construct a collection $\{ U_p : p \in [0,1] \cap \mathbb{Q} \}$ of open sets. Let $Q = [0,1] \cap \mathbb{Q}$. Recalling that Q is countable, we enumerate it as $Q = \{ p_n : n \in \mathbb{N} \}$, and for convenience we will assume that $p_0 = 1$ and $p_1 = 0$.

To be more clear, we are going to define by induction (on Q, or more precisely on the indices of Q) a collection $\{ U_p : p \in Q \}$ of open subsets with the property:

$$p < q \Longrightarrow \overline{U_p} \subseteq U_q. \tag{$\star$}$$

First, let $U_1 = X \setminus D$. Next, using the fact that $(X, \mathcal{T})$ is normal, let U_0 be an open set such that

$$C \subseteq U_0 \subseteq \overline{U_0} \subseteq U_1.$$

This is the base of our induction, and obviously satisfies $(\star)$.

Now suppose we have defined U_{p_k} for all $k = 0, 1, 2, \ldots, n$ satisfying $(\star)$, and we will set about defining $U_{p_{n+1}}$. Just to simplify our notation, let $P = \{p_1, p_2, \ldots, p_n\}$, and let $r = p_{n+1}$.

First, we situate the number r in the set P. That is, P is a finite collection of rational numbers in $[0,1]$, and moreover P contains 0 and 1 by construction. This means that if we think of $P \cup \{r\}$ as a linear order (with the usual ordering of rationals), r has an immediate predecessor and an immediate successor in P. Call these two elements p and q, respectively. (For example, if $P = \{0, \frac{1}{3}, \frac{2}{3}, 1\}$ and $r = \frac{5}{6}$, we would have $p = \frac{2}{3}$ and $q = 1$.)

Now by the inductive hypothesis, we have that $\overline{U_p} \subseteq U_q$. Using normality, find an open set U_r such that

$$\overline{U_p} \subseteq U_r \subseteq \overline{U_r} \subseteq U_q.$$

With this defined, the collection $U_{p_1}, U_{p_2}, \ldots, U_{p_n}, U_r = U_{p_{n+1}}$ satisfies $(\star)$. This completes the induction.

What we just did was actually quite simple. At each stage of the induction, we had previously dealt with finitely many elements of Q, the set of which we called P, and we are handed a new element r of Q. We found the pair of numbers p and q in P that r fit exactly between, which we could always do since r could not be 0 or 1, and used the alternative characterization of normality to fit an open set in between the two open sets U_p and U_q (though we actually asked for slightly more: $U_p \subseteq \overline{U_p} \subseteq U_r \subseteq \overline{U_r} \subseteq U_q$).

By the end of the induction, we have created a collection of open sets $\{ U_p : p \in Q \}$ whose ordering under inclusion exactly mirrors the ordering of Q as a set of rationals.

At this point, the reader is **strongly** encouraged to draw a picture. Imagine that Q is enumerated in a straightforward way, like $Q = \{1, 0, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \ldots\}$, and draw out what happens in the first few steps of the induction.

The next step is to extend our collection of open sets to the whole set $\mathbb{Q}$ of rational numbers. We have defined U_p for all $p \in [0,1] \cap \mathbb{Q}$. Now if $p \in (-\infty, 0) \cap \mathbb{Q}$, let $U_p = \emptyset$. If $p \in (1, \infty) \cap \mathbb{Q}$, let $U_p = X$. Note that the newly defined collection $\{ U_p \,:\, p \in \mathbb{Q} \}$ still satisfies the property $(\star)$:

$$p < q \Longrightarrow \overline{U_p} \subseteq U_q.$$

We now set about defining our function, which will defined in almost exactly the same way as the example function g in the previous section was defined.

For each $x \in X$, define

$$\mathbb{Q}(x) := \{ p \in \mathbb{Q} \,:\, x \in U_p \} .$$

Note first that $\mathbb{Q}(x)$ is bounded below by 0, since if p is negative $U_p = \emptyset$ and so $x \notin U_p$. Then note that $\mathbb{Q}(x) \neq \emptyset$, since $x \in X = U_p$ for all $p > 1$. In other words, for every $x \in X$, $\mathbb{Q}(x)$ contains every rational number larger than 1, no rational numbers less than zero, and some rational numbers in between.

This means that for all $x \in X$, $\mathbb{Q}(x)$ has a greatest lower bound, and this is what we define to be the value of our function. Define $f : X \to [0,1]$ by

$$f(x) = \inf \mathbb{Q}(x) = \inf \{ p \in \mathbb{Q} \,:\, x \in U_p \} .$$

This function is the "interesting idea" in this proof. From here on out we will do the grunt work of showing that this function works.

Take a moment to convince yourself that $f(x) \in [0,1]$ for all $x \in X$. After that, it remains to show that f is continuous, and that it separates C and D.

Claim. *f separates C and D. That is $f(x) = 0$ for all $x \in C$ and $f(x) = 1$ for all $x \in D$.*

Proof. Let $x \in C$. Then $x \in C \subseteq U_0 \subseteq U_p$ for all $p \geq 0$. Therefore $\mathbb{Q}(x) = [0, \infty) \cap \mathbb{Q}$, and so $f(x) = 0$.

On the other hand, let $x \in D$. Then $x \notin U_1$, and so $x \notin U_p$ for all $p \leq 1$. Therefore $\mathbb{Q}(x) = (1, \infty) \cap \mathbb{Q}$, and so $f(x) = 1$. □

That was easy. The tricky part is showing that f is continuous. Before starting on that, we prove two simple facts about f which we will use several times.

Claim.

1. *If $x \in \overline{U_p}$, then $f(x) \leq p$.*
2. *If $x \notin U_p$, then $f(x) \geq p$.*

Proof. (1) Suppose $x \in \overline{U_p}$. (Note that this does not imply that $x \in U_p$, which would make the proof easy.) Then by construction $x \in \overline{U_p} \subseteq U_q$ for all rationals $q > p$, and therefore $(p, \infty) \cap \mathbb{Q}$ is a subset of $\mathbb{Q}(x)$. This implies that $\inf \mathbb{Q}(x) \leq p$, as required.

(2) Suppose $x \notin U_p$. Then by construction $x \notin U_q$ for any $q \leq p$. In other words, $(-\infty, p] \cap \mathbb{Q}(x) = \emptyset$. This implies that $\inf \mathbb{Q}(x) \geq p$. □

With these easy facts out of the way...

Claim. *f is continuous.*

Proof. Suppose $U = (a, b)$ is an open interval in $\mathbb{R}$ that intersects $[0, 1]$ (so that $(a, b) \cap [0, 1]$ is a basic open subset in the subspace topology). We want to show that $f^{-1}(U)$ is open in X. To do this, we will fix an arbitrary point $x \in f^{-1}(U)$, and find an open set V of X such that $x \in V \subseteq f^{-1}(U)$, or in other words such that $f(x) \in f(V) \subseteq U$.

So fix an $x \in f^{-1}(U)$ as we said. Then $f(x) \in (a, b)$, so we can find rational numbers p and q such that

$$a < p < f(x) < q < b.$$

Since $p < f(x)$, it follows from the contrapositive of part (1) of the previous claim that $x \notin \overline{U_p}$. On the other hand, since $f(x) < q$, it follows from the contrapositive of part (2) of the previous claim that $x \in U_q$. From these two conclusions combined it follows that $x \in U_q \setminus \overline{U_p}$. This will be our open set V.

It remains to show that $f(V) \subseteq (a, b)$. We will show this by picking a point $y \in V$ and showing that $f(y) \in (a, b)$. So fix $y \in V$. Then by definition $y \in U_q \subseteq \overline{U_q}$, and therefore $f(y) \leq q < b$ by part (1) of the previous claim. On the other hand, since $y \notin \overline{U_p} \supseteq U_p$, we have that $f(y) \geq p > a$ by part (2) of the previous claim. Therefore

$$f(y) \in [p, q] \subseteq (a, b),$$

as required. □

This completes the proof of Urysohn's Lemma. We did it!

5 The Tietze Extension Theorem

This is the first important consequence of Urysohn's Lemma that we will see. We will not prove it, because the proof is quite long and, while interesting, it is not an efficient use of our time. This theorem is about extending continuous functions from subsets of a topological space to the whole topological space. Its applications to other areas of topology are numerous, but we will not deal with many of them in this class. We will mention one such application after stating the theorem, and put off its proof until a bit later in the course after we have defined all the terms involved.

Theorem 5.1 (Tietze Extension Theorem)**.** *Let* $(X, \mathcal{T})$ *be normal, and let* $A \subseteq X$ *be closed. Then any continuous function from* A *to the closed interval* $[a, b] \subseteq \mathbb{R}$ *(or to* $\mathbb{R}$*) can be extended to a continuous function from all of* X *to* $[a, b]$ *(or, respectively, to* $\mathbb{R}$*).*

Here is one application to a familiar setting. Recall the Extreme Value Theorem from first year calculus:

Theorem 5.2 (Extreme Value Theorem)**.** *Suppose* $f : [a, b] \to \mathbb{R}$ *is a continuous function. Then* f *achieves a minimum and a maximum value.*

This result is often mentioned in second year calculus in the following, more general form: Let $C \subseteq \mathbb{R}^n$ be compact. Then any continuous function $f : C \to \mathbb{R}$ achieves a minimum and maximum value.

On a sufficiently nice topological space, the Tietze Extension Theorem allows us to turn this behaviour into a *characterization* of compactness.

Theorem 5.3. *A metrizable space* X *is compact if and only if every continuous function* $f : X \to \mathbb{R}$ *is bounded.*

We have not yet defined what "compact" means in this course, but rest assured we will prove this result when we are able to do so.