

# The countable chain condition in products

## Contents

<b>1</b>	<b>Motivation</b>	<b>2</b>
<b>2</b>	<b>A partial result about the productivity of the ccc</b>	<b>2</b>
2.1	Preliminaries . . . . .	2
2.2	Results . . . . .	2
<b>3</b>	<b>Martin's Axiom implies that the ccc is productive</b>	<b>5</b>
3.1	Preliminaries . . . . .	5
3.2	Results . . . . .	6

## 1 Motivation

We have learned that the countable chain condition behaves unusually with respect to products. In fact, we learned that the question of whether the ccc is finitely productive is independent of the usual ZFC axioms. This means that given an arbitrary collection of ccc spaces, there is no hope of deciding whether their product is ccc.

With that said, there are still some partial results that are available to us. The first section of this note will outline the most important one of these, and some of its corollaries.

The next sections of this note will outline, in relatively simple terms, why the question of whether the ccc is finitely productive is independent of the usual ZFC axioms.

## 2 A partial result about the productivity of the ccc

### 2.1 Preliminaries

Recall the following definitions and facts from throughout the course.

**Definition 2.1.** A topological space  $(X, \mathcal{T})$  is said to have the countable chain condition if any collection of pairwise disjoint, nonempty, open subsets of  $X$  is countable.

**Notation.** Given a set  $A$ , we denote by  $[A]^{<\omega}$  the set of all finite subsets of  $A$ . More formally:

$$[A]^{<\omega} := \{ F \subseteq A : F \text{ is finite} \}.$$

**Definition 2.2.** Let  $A$  be a set. A collection  $\mathcal{A} \subseteq [A]^{<\omega}$  is called a  $\Delta$ -system if there is a finite set  $r \in [A]^{<\omega}$  (which may be empty) such that  $a \cap b = r$  for all distinct  $a, b \in \mathcal{A}$ . The set  $r$  is called the root of the  $\Delta$ -system.

**Theorem 2.3** ( $\Delta$ -System Lemma). Let  $A$  be an uncountable set, and let  $\mathcal{A} \subseteq [A]^{<\omega}$  be an uncountable collection of finite subsets of  $A$ . Then there is an uncountable  $\mathcal{B} \subseteq \mathcal{A}$  that is a  $\Delta$ -system.

### 2.2 Results

**Theorem 2.4.** Let  $I$  be a nonempty indexing set, and let  $\{(X_\alpha, \mathcal{T}_\alpha) : \alpha \in I\}$  be a family of topological spaces such that  $\prod_{\alpha \in F} X_\alpha$  has the countable chain condition for all  $F \in [I]^{<\omega}$ . Then  $\prod_{\alpha \in I} X_\alpha$  has the countable chain condition.

*Proof.* Let  $X = \prod_{\alpha \in I} X_\alpha$  be the product of the given spaces, with its product topology. The theorem is tautological if  $I$  is finite. We first treat the case where  $I$  is uncountable.

Assume for the sake of contradiction that  $\mathcal{U}$  is an uncountable collection of pairwise disjoint, nonempty, open subsets of  $X$ . We may assume without loss of generality that each of the sets in  $\mathcal{U}$  are basic open subsets of the product topology.

By definition of the product topology, for each  $U = \prod_{\alpha \in I} U_\alpha \in \mathcal{U}$ , there is a finite set  $F_U \subseteq I$  such that  $U_\alpha \neq X_\alpha$  if and only if  $\alpha \in F_U$ .

First, we present a useful lemma:

**Lemma 2.5.** *Let  $U = \prod_{\alpha \in I} U_\alpha$  and  $V = \prod_{\alpha \in I} V_\alpha$  be disjoint, nonempty, basic open subsets of  $X$ . Then  $F_U \cap F_V \neq \emptyset$ , and moreover there is some  $\alpha \in F_U \cap F_V$  such that  $U_\alpha \cap V_\alpha = \emptyset$ .*

*Proof.* Assume  $U$  and  $V$  are disjoint and nonempty. First, suppose for the sake of contradiction that  $F_U \cap F_V = \emptyset$ . Then we can pick...

- ...a point  $a_\alpha \in U_\alpha$  for all  $\alpha \in F_U$ ,
- ...a point  $b_\alpha \in V_\alpha$  for all  $\alpha \in F_V$ , and
- ...a point  $c_\alpha \in X_\alpha$  for all  $\alpha \in I \setminus (F_U \cup F_V)$ .

Then the point  $x \in X$  defined by:

$$\pi_\alpha(x) = \begin{cases} a_\alpha & \alpha \in F_U \\ b_\alpha & \alpha \in F_V \\ c_\alpha & \text{otherwise} \end{cases}$$

is in  $U \cap V$ , contradicting the assumption that  $U$  and  $V$  are disjoint.

Next, let  $F = F_U \cap F_V$  and suppose for the sake of contradiction that  $U_\alpha \cap V_\alpha \neq \emptyset$  for all  $\alpha \in F$ . Then we can pick...

- ...a point  $a_\alpha \in U_\alpha \cap V_\alpha$  for all  $\alpha \in F$ ,
- ...a point  $b_\alpha \in U_\alpha$  for all  $\alpha \in F_U \setminus F$ ,
- ...a point  $c_\alpha \in V_\alpha$  for all  $\alpha \in F_V \setminus F$ , and
- ...a point  $d_\alpha \in X_\alpha$  for all  $\alpha \in I \setminus (F_U \cup F_V)$ .

Then the point  $x \in X$  defined by

$$\pi_\alpha(x) = \begin{cases} a_\alpha & \alpha \in F \\ b_\alpha & \alpha \in F_U \setminus F \\ c_\alpha & \alpha \in F_V \setminus F \\ d_\alpha & \text{otherwise} \end{cases}$$

is in  $U \cap V$ , again contradicting the assumption that  $U \cap V = \emptyset$ . This completes the proof of the lemma.  $\square$

Getting back to the proof of the theorem, apply the  $\Delta$ -system lemma to the collection  $\mathcal{A} = \{F_U : U \in \mathcal{U}\}$  to find an uncountable  $\Delta$ -system  $\mathcal{B} \subseteq \mathcal{A}$ . Let  $r$  be the root of this  $\Delta$ -system.

**Claim.**  $r \neq \emptyset$ .

*Proof.* Suppose for the sake of contradiction that  $r = \emptyset$ . Given two sets  $F, G \in \mathcal{B}$ , let  $U$  and  $V$  be elements of  $\mathcal{U}$  such that  $F_U = F$  and  $F_V = G$  (we have to say this since the map  $U \mapsto F_U$  need not be injective). Since  $r = \emptyset$ , this means  $F \cap G = F_U \cap F_V = \emptyset$ , which would in turn imply that  $U \cap V \neq \emptyset$  by the lemma, contradicting the assumption that  $\mathcal{U}$  consists of pairwise disjoint sets. This finishes the proof of the claim.  $\square$

Let  $\mathcal{U}' = \{U \in \mathcal{U} : F_U \in \mathcal{B}\}$ . For each  $U = \prod_{\alpha \in I} U_\alpha \in \mathcal{U}'$ , define:

$$p(U) = \prod_{\alpha \in r} U_\alpha.$$

Finally, let  $\mathcal{V} = \{p(U) : U \in \mathcal{U}'\}$ . So  $\mathcal{V}$  is a collection of open subsets of  $\prod_{\alpha \in r} X_\alpha$ , a space that by hypothesis is ccc, since  $r$  is finite.

**Claim.** *The map  $\mathcal{U}' \rightarrow \mathcal{V}$  given by  $U \mapsto p(U)$  is injective.*

*Proof.* Let  $U = \prod_{\alpha \in I} U_\alpha$  and  $V = \prod_{\alpha \in I} V_\alpha$  be given elements of  $\mathcal{U}'$ , and note that by definition of  $\mathcal{U}'$ , we have that  $F_U \cap F_V = r$ .

Then if  $U \neq V$  and  $p(U) = p(V)$ , we would have that  $U_\alpha = V_\alpha$  for all  $\alpha \in r = F_U \cap F_V$ . By the lemma, this would imply that  $U \cap V \neq \emptyset$ , contradicting the assumption that  $\mathcal{U}$  consists of pairwise disjoint sets.  $\square$

It follows from the last claim that  $\mathcal{V}$  is uncountable, since  $\mathcal{U}'$  is uncountable. By hypothesis,  $\prod_{\alpha \in r} X_\alpha$  is ccc, and so in particular there must exist  $p(U), p(V) \in \mathcal{V}$  such that  $p(U) \cap p(V) \neq \emptyset$ . But then, by the lemma again, we could find a point in  $X$  inside  $U \cap V$ , again contradicting the assumption that  $\mathcal{U}$  consists of disjoint sets. Therefore, the whole product space  $X$  must have the countable chain condition.

Finally, we treat the case in which  $I$  is countable, which is much simpler than the previous case. Again, assume for the sake of contradiction that  $\mathcal{U}$  is an uncountable collection of pairwise disjoint, nonempty, open subsets of  $X$ . For each  $U \in \mathcal{U}$ , let  $F_U \subseteq I$  be as in the previous case. There are only countably many finite subsets of  $I$ , and therefore by the pigeonhole principle there must be a finite  $F \subseteq I$  and an uncountable  $\mathcal{U}' \subseteq \mathcal{U}$  such that  $F_U = F$  for all  $U \in \mathcal{U}'$ . In other words,  $\pi_\alpha(U) = X_\alpha$  for all  $\alpha \in I \setminus F$  and all  $U \in \mathcal{U}'$ .

We are now in essentially the same situation as in the previous proof (but now with  $F$  instead of  $r$ ). For each  $U = \prod_{\alpha \in I} U_\alpha \in \mathcal{U}'$ , let  $p(U) = \prod_{\alpha \in F} U_\alpha$ . Then the mapping  $U \mapsto p(U)$  is injective, and therefore since  $\prod_{\alpha \in F} X_\alpha$  is ccc by hypothesis, there must exist  $U, V \in \mathcal{U}'$  such

that  $p(U) \cap p(V) \neq \emptyset$ . But then, by the lemma, this implies that  $U \cap V \neq \emptyset$ , contradicting the assumption that  $\mathcal{U}$  consists of pairwise disjoint sets.  $\square$

It follows easily from the theorem that if  $\phi$  is a topological property such that:

- $\phi$  is finitely productive;
- $\phi$  implies the ccc,

Then any product of topological spaces with  $\phi$  will also have the ccc. In particular:

**Corollary 2.6.** *Let  $\{(X_\alpha, \mathcal{T}_\alpha) : \alpha \in I\}$  be a family of separable spaces. Then their product  $\prod_{\alpha \in I} X_\alpha$  has the countable chain condition.*

This result allows us to easily construct ccc, non-separable spaces. For example, if  $I$  is an indexing set with cardinality strictly larger than  $\mathbb{R}$ , then  $\{0, 1\}^I$  is ccc but not separable.

Another notable corollary:

**Corollary 2.7.** *If, under some additional set theoretic assumptions, we can prove that the ccc is finitely productive, we will also have shown that, under those assumptions, the ccc is arbitrarily productive.*

In the next section, we will use this corollary to prove that under a certain additional set theoretic assumption the ccc is arbitrarily productive.

### 3 Martin's Axiom implies that the ccc is productive

Martin's Axiom (MA) is a statement that is independent of the usual ZFC axioms of set theory. It's a very useful tool for proving many things, including some independence results. It has many forms, but by far the most useful way to think about it is as a statement about partial orders.

We have already seen what a partial order is, but we will need a few more definitions before we can see the statement of Martin's Axiom. Some of these we have seen before, and some of them are partial order versions of things we have seen in a topological context.

#### 3.1 Preliminaries

**Definition 3.1.** *Let  $(\mathbb{P}, \leq)$  be a partial order.*

- Two elements  $p, q \in \mathbb{P}$  are called compatible if there is an element  $r \in \mathbb{P}$  such that  $r \leq p$  and  $r \leq q$ . If they are not compatible they are called incompatible. A subset  $A \subseteq \mathbb{P}$  is called an antichain if every pair of elements of  $A$  are incompatible.

- $(\mathbb{P}, \leq)$  is said to have the countable chain condition (or ccc) if every antichain in  $\mathbb{P}$  is countable.
- A subset  $D \subseteq \mathbb{P}$  is called dense in  $\mathbb{P}$  (or simply dense if the partial order in question is clear from context) if for every  $p \in \mathbb{P}$  there is a  $d \in D$  such that  $d \leq p$ .
- A strict, non-empty subset  $\mathcal{F} \subseteq \mathbb{P}$  is called a filter on  $\mathbb{P}$  if the following conditions hold:
  1.  $\mathcal{F}$  is closed upwards. In other words, if  $p \in \mathcal{F}$  and  $p \leq q$ , then  $q \in \mathcal{F}$ .
  2.  $\mathcal{F}$  is directed. In other words, if  $p, q \in \mathcal{F}$ , then there is a  $r \in \mathcal{F}$  such that  $r \leq p$  and  $r \leq q$ . Another way to say this is that every two elements of  $\mathcal{F}$  are compatible, and that this compatibility is witnessed by an element of  $\mathcal{F}$ .

With these definitions in place, we can state Martin's Axiom. To be a little more precise, we are going to state what's usually called  $\text{MA}(\omega_1)$ .

Let  $(\mathbb{P}, \leq)$  be a partial order with the countable chain condition, and let  
 $(\text{MA}(\omega_1)) \quad \{D_\alpha : \alpha \in \omega_1\}$  be a collection of non-empty, dense subsets of  $\mathbb{P}$ . Then  
 there is a filter  $\mathcal{G}$  on  $\mathbb{P}$  such that  $\mathcal{G} \cap D_\alpha \neq \emptyset$  for all  $\alpha \in \omega_1$ .

This statement is independent of the usual axioms of ZFC (the proof of this is, sadly, well beyond the scope of this note and our course). In other words, assuming that this statement is true cannot lead to a contradiction, and neither can assuming it is not true. It is with the help of this and many other statements like this that mathematicians can prove certain statements are consistent with ZFC.

## 3.2 Results

We will now prove that under the assumption that  $(\text{MA}(\omega_1))$  is true, that any product of ccc topological spaces is ccc. We start with a preliminary lemma.

**Lemma 3.2.** *Assume  $(\text{MA}(\omega_1))$ .*

*Let  $(X, \mathcal{T})$  be a ccc topological space, and suppose  $\{U_\alpha : \alpha \in \omega_1\}$  is a collection of non-empty open subsets of  $X$ . Then there is an uncountable  $A \subseteq \omega_1$  such that for any  $\alpha, \beta \in A$ ,  $U_\alpha \cap U_\beta \neq \emptyset$ .*

*Proof.* For each  $\alpha \in \omega_1$ , define

$$V_\alpha = \bigcup_{\beta > \alpha} U_\beta.$$

These new open sets form a decreasing chain of sets, in the sense that if  $\alpha < \beta$ , then  $V_\beta \subseteq V_\alpha$ .

We first show that this chain “stabilizes” at some point. More specifically, that there is an  $\alpha \in \omega_1$  such that for all  $\beta > \alpha$ ,  $\overline{V_\beta} = \overline{V_\alpha}$ . Indeed, if no such  $\alpha$  exists, then for every  $\beta \in \omega_1$  we can find a  $\gamma_\beta > \beta$  such that  $V_\beta \setminus \overline{V_{\gamma_\beta}} \neq \emptyset$ . Any two sets of this form are disjoint, and

so proceeding this way we can construct an unbounded (and therefore uncountable) subset of  $\omega_1$  and a corresponding uncountable collection of disjoint open subsets of  $X$ , contradicting the assumption that  $X$  has the ccc.

So, fix an  $\alpha \in \omega_1$  as in the previous paragraph. Consider the partial order  $(\mathbb{P}, \subseteq)$ , where

$$\mathbb{P} = \{ P \subseteq V_\alpha : P \text{ is open and non-empty} \}.$$

$\mathbb{P} \subseteq \mathcal{T}$ , and so  $\mathbb{P}$  has the ccc (as a partial order) since  $(X, \mathcal{T})$  has the ccc (as a topological space).

We want to apply  $(\text{MA}(\omega_1))$  to  $\mathbb{P}$ , and so we need some dense subsets of  $\mathbb{P}$  to work with.

For each  $\beta \in \omega$ , define a set  $D_\beta \subseteq \mathbb{P}$  by:

$$D_\beta = \{ P \in \mathbb{P} : P \subseteq U_\gamma \text{ for some } \gamma > \beta \}.$$

**Claim.** For all  $\beta \in \omega_1$ ,  $D_\beta$  is dense in  $\mathbb{P}$ .

*Proof.* Fix a  $\beta \in \omega_1$ . By definition of  $\alpha$ , we have that  $\overline{V_\alpha} \subseteq \overline{V_\beta}$  (if  $\alpha < \beta$  then in fact we have  $\overline{V_\alpha} = \overline{V_\beta}$ , while if  $\beta < \alpha$  then this follows from our early observation that these sets form a decreasing chain).

Fix an arbitrary  $P \in \mathbb{P}$ . We want to find a  $Q \in D_\beta$  such that  $Q \subseteq P$ . From our observation just above, it follows that  $P \cap V_\beta \neq \emptyset$ . In other words, recalling the definition of  $V_\beta$ :

$$P \cap \left( \bigcup_{\gamma > \beta} U_\gamma \right) \neq \emptyset,$$

whence it follows that  $P \cap U_\gamma \neq \emptyset$  for some  $\gamma > \beta$ . Let  $Q = P \cap U_\gamma$ . Then, to summarize:

- $Q$  is open, and  $Q \subseteq P$ .
- $Q \subseteq U_\gamma$  and  $\gamma > \beta$ , and so  $Q \in D_\beta$ .

This completes the proof of the claim. □

Finally, we are in a position to apply  $(\text{MA}(\omega_1))$ . Indeed, let  $\mathcal{G} \subseteq \mathbb{P}$  be a filter such that

$$\mathcal{G} \cap D_\beta \neq \emptyset \quad \text{for all } \beta \in \omega_1.$$

We will define the subset  $A \subseteq \omega_1$  that we are looking for from  $\mathcal{G}$  in the following way:

$$A := \{ \gamma \in \omega_1 : P \subseteq U_\gamma \text{ for some } P \in \mathcal{G} \}.$$

**Claim.**  $A$  is uncountable.

*Proof.* We will show that  $A$  is unbounded in  $\omega_1$ , from which it follows that it is uncountable. Indeed, fix an arbitrary  $\beta \in \omega_1$ . By construction,  $\mathcal{G} \cap D_\beta \neq \emptyset$ , so let  $P$  be an element of this intersection. By definition of  $D_\beta$ , this means there is a  $\gamma > \beta$  such that  $P \subseteq U_\gamma$ . But then  $\gamma \in A$ , completing the proof of the claim. □

**Claim.** For any  $\alpha, \beta \in A$ ,  $U_\alpha \cap U_\beta \neq \emptyset$ .

*Proof.* This follows immediately from the fact that  $\mathcal{G}$  is a filter, and is therefore directed. Indeed, let  $\alpha, \beta \in A$ . By definition of  $A$ , this means there are  $P_\alpha, P_\beta \in \mathcal{G}$  such that  $P_\alpha \subseteq U_\alpha$  and  $P_\beta \subseteq U_\beta$ . But  $\mathcal{G}$  is a filter, and so  $P_\alpha \cap P_\beta \neq \emptyset$ , from which it follows that  $U_\alpha \cap U_\beta \neq \emptyset$ .  $\square$

$\square$

The preceding claim is a bit technical, but having proved it we can now get the result we want very easily.

**Theorem 3.3.** Assume  $(MA(\omega_1))$ .

Then any product of ccc topological spaces is ccc.

*Proof.* By Corollary 2.6, it suffices to show that the product of any two ccc spaces is ccc.

Let  $X$  and  $Y$  be ccc topological spaces, and assume for the sake of contradiction that  $\{W_\alpha : \alpha \in \omega_1\}$  is a collection of pairwise disjoint, non-empty open subsets of  $X \times Y$ . Shrinking each one if necessary, we may assume without loss of generality that  $W_\alpha = U_\alpha \times V_\alpha$ , where  $U_\alpha \subseteq X$  and  $V_\alpha \subseteq Y$  are open sets.

Applying the lemma to the collection  $\{U_\alpha : \alpha \in \omega_1\}$ , we find an uncountable  $A \subseteq \omega_1$  such that for any  $\alpha, \beta \in A$ ,  $U_\alpha \cap U_\beta \neq \emptyset$ . Fix arbitrary  $\alpha \neq \beta \in A$ . Then  $U_\alpha \cap U_\beta \neq \emptyset$  as we just said, but on the other hand  $(U_\alpha \times V_\alpha) \cap (U_\beta \times V_\beta) = W_\alpha \cap W_\beta$  is empty by assumption. This forces  $V_\alpha \cap V_\beta = \emptyset$ . Since  $\alpha$  and  $\beta$  were arbitrary, this means  $\{V_\alpha : \alpha \in \omega_1\}$  is a pairwise disjoint collection of non-empty open subsets of  $Y$ , contradicting the assumption that  $Y$  has the ccc.  $\square$