Things You Should Know

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1 Basic Set Theory

I will assume students are familiar with all of these terms and symbols. Please ask about anything that seems unfamiliar or unclear from their definitions here.

In the following, A, B, X, Y are sets. I is an indexing set, $\{ A_\alpha : \alpha \in I \}$ and $\{ B_\alpha : \alpha \in I \}$ are families of sets indexed by I, and $\mathcal{C}$ is a collection of sets.

- Empty set: $\emptyset$, the set with no elements.
- Subset: $A \subseteq B$ means "$x \in A \implies x \in B$"
- Power set: $\mathcal{P}(X) \coloneqq \{ A : A \subseteq X \}$
- Union: $A \cup B \coloneqq \{ x : x \in A \text{ or } x \in B \}$
- Intersection: $A \cap B \coloneqq \{ x : x \in A \text{ and } x \in B \}$
- Complement: If $A \subseteq X$, then $X \setminus A \coloneqq \{ x : x \in X \text{ and } x \notin A \}$
- Indexed union: $\bigcup_{\alpha \in I} A_\alpha \coloneqq \{ x : \exists \alpha \in I,\ x \in A_\alpha \}$
- Non-indexed union: $\bigcup \mathcal{C} \coloneqq \bigcup_{X \in \mathcal{C}} X$.
- Indexed intersection: $\bigcap_{\alpha \in I} A_\alpha \coloneqq \{ x : \forall \alpha \in I,\ x \in A_\alpha \}$
- Non-indexed intersection: $\bigcap \mathcal{C} \coloneqq \bigcap_{X \in \mathcal{C}} X$.
- Cartesian product of two sets: $X \times Y \coloneqq \{ (x, y) : x \in X,\ y \in Y \}$

2 Functions

In the following, let X and Y be sets, and let $f : X \to Y$ be a function.

- X is the <u>domain</u> of f.
- Y is the <u>codomain</u> of f.
- $f(X) = \{ f(x) : x \in X \} \subseteq Y$ is the <u>range</u> or <u>image</u> of f.

- f is injective (or one-to-one, or an injection)

$$\forall a, b \in X, \quad f(a) = f(b) \implies a = b.$$

- f is surjective (or onto, or a surjection) if its range is its entire codomain.
- f is bijective (or a bijection) if it is both injective and a surjective.
- The composition of two injective functions is again injective.
- The composition of two surjective functions is again surjective.
- The composition of two bijective functions is again bijective.
- Given a subset $B \subseteq Y$, the preimage of B is the set $f^{-1}(B) := \{\, x \in X \,:\, f(x) \in B \,\}$.
- If f is an injection with range Y, then its inverse function $f^{-1} : Y \to X$ is (1) a function; and (2) injective.

3 DeMorgan's Laws and other interactions

The following two expressions are generalized versions of what are called De Morgan's Laws. They describe how unions and intersections interact with complementation.

- $X \setminus \left(\bigcup_{\alpha \in I} A_\alpha \right) = \bigcap_{\alpha \in I} (X \setminus A_\alpha)$
- $X \setminus \left(\bigcap_{\alpha \in I} A_\alpha \right) = \bigcup_{\alpha \in I} (X \setminus A_\alpha)$

The following are elementary facts about how functions interact with operations on subsets of their domains, codomains and ranges. Throughout the following, let X and Y be sets, let $f : X \to Y$ be a function, and let $A, B \subseteq X$ and $C, D \subseteq Y$.

- $A \subseteq B$ implies $f(A) \subseteq f(B)$
- $C \subseteq D$ implies $f^{-1}(C) \subseteq f^{-1}(D)$
- $f(A \cup B) = f(A) \cup f(B)$
- $f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D)$
- $f(A \cap B) \subseteq f(A) \cap f(B)$
- $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$

- $f(A) \setminus f(B) \subseteq f(A \setminus B)$
- $f^{-1}(C \setminus D) = f^{-1}(C) \setminus f^{-1}(D)$
- $f(X \setminus f^{-1}(Y \setminus C)) \subseteq C$
- $A \subseteq f^{-1}(f(A))$, (with equality if f is injective)
- $f(f^{-1}(C)) \subseteq C$, (with equality if f is surjective)
- $f^{-1}(Y \setminus C) = X \setminus f^{-1}(C)$

4 Countability

We will spend some time on this in class, but I do expect these words to be familiar to you.

Definition 1. *A set A is said to be <u>countably infinite</u> if there exists a bijection $f : \mathbb{N} \to A$. A set A is said to be <u>countable</u> if it is finite or countably infinite. If A is infinite but not countably infinite, A is said to be <u>uncountable</u>.*

The following theorem gives some equivalent conditions for being countable:

Theorem 2. *For an infinite set A, the following are equivalent:*

1. *A is countable.*
2. *There is an injection $f : A \to \mathbb{N}$.*
3. *There is a surjection $g : \mathbb{N} \to A$.*

Fact: The following sets are countable:

- $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$, the set of algebraic numbers.
- Any infinite subset of a countable set.
- The Cartesian product of two countable sets (and, inductively, the Cartesian product of a finite number of countable sets).
- The union of finitely many countable sets.
- The union of a countable collection of countable sets.
- The countable union of some countable sets and some finite sets.

Fact: The following sets are uncountable:

- $\mathbb{R}$, $\mathbb{R} \setminus \mathbb{Q}$ (the irrational numbers), the set of non-algebraic numbers (i.e. the set of transcendental numbers), $\mathbb{R}^n$.
- Any superset of an uncountable set.
- The power set of any infinite set (countable or otherwise), e.g. $\mathcal{P}(\mathbb{N})$.
- The set $\mathbb{N}^{\mathbb{N}}$ of functions from $\mathbb{N}$ to $\mathbb{N}$.

The following are two very useful combinatorial facts

Theorem 3 (Infinite Pigeonhole Principle). *Let X be an infinite set, and A a finite set. If $c : X \to A$ is a function, then there is an $a \in A$ such that $c^{-1}(a)$ is infinite.*

Theorem 4 (Uncountable Pigeonhole Principle). *Let X be an uncountable set, and A a countable set. If $c : X \to A$ is a function, then there is an $a \in A$ such that $c^{-1}(a)$ is uncountable.*

The latter theorem can be restated in plain English as "*If you try to put uncountably many pigeons into countably many holes, then there is a hole with uncountably many pigeons*".

5 Selected basic facts about $\mathbb{R}$

First recall: $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$. (For us, $0 \notin \mathbb{N}$.)

Fact: Between any two distinct real numbers:

- There are infinitely many rational numbers.
- There are infinitely many irrational numbers.

Fact: Here are some useful facts from calculus:

- $\bigcup_{n \in \mathbb{N}} [\frac{1}{n}, 1] = (0, 1]$.
- $\bigcup_{n \in \mathbb{N}} [0, n] = [0, \infty)$.
- $\sum_{n \in \mathbb{N}} 2^{-n} = 1$.